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Few General Identities of Rogers-Ramanujan Type

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Abstract

In this paper, we have derived some identities of Rogers-Ramanujan Type modulo $9s$, $11s$, $15s$ and $45s$ (where s is any finite positive integer) by using the transformation theory of Basic Hypergeometric Series and Bailey Lemma. Some particular cases of such identities also have been derived.

Keywords: Rogers-Ramanujan Identity, Basic Hypergeometric Series, Jacobi's Triple Product Identity, Bailey's Lemma.

1. Introduction

The most famous of the "Series = Product" Identities are:

For |q|<1,

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n}, n\not\equiv 0,\pm 2 \pmod 5$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n}, n\not\equiv 0,\pm 1 \pmod 5$$

Where

$$(q;q)_n=(1-q)\left(1-q^2\right)\ldots\left(1-q^n\right),$$

Which are known as the celebrated original Rogers-Ramanujan Identities and they have motivated extensive research work over the past hundred years.

In the last century, W. N. Bailey [4], G.E. Andrews [3], A. Verma and V.K Jain [1], and many others have extensively used the Transformation Theory of Basic Hypergeometric Series to derive many Identities of Rogers Ramanujan Type.In this paper, we use the techniques laid down by A. Verma and V.K Jain to derive some more identities of Rogers-Ramanujan Type which are not listed in [1].

Throughout this paper, we assume $|\mathbf{q}| < \mathbf{1}$ and, as customary, we define

$$(a;q)_0=1$$
$$(a;q)_n=\prod_{k=0}^{n-1}(1-aq^k), \text{ for } n\geq 1$$
$$\text{and } (a;q)_\infty=\prod_{k=1}^{\infty}(1-aq^k).$$

It follows that

$$(a;q)_n=\frac{(a;q)_\infty}{(aq^n;q)_\infty}$$

The multiple q −shifted factorials is defined by

$$(a_1,a_2,\ldots\ldots a_m;q)_n=(a_1;q)_n(a_2;q)_n\ldots.(a_m;q)_n$$
$$(a_1,a_2,\ldots\ldots a_m;q)_\infty=(a_1;q)_\infty(a_2;q)_\infty\ldots.(a_m;q)_\infty$$

The Basic Hyper geometric Series is

$$ {}_{p+1}\phi_{p+r}\left(\begin{matrix} a_1,a_2,\ldots\ldots,a_{p+1};q;x \\ b_1,b_2,\ldots\ldots,b_{p+r} \end{matrix}\right) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \ldots (a_{p+1};q)_n x^n (-1)^{nr} q^{\frac{n(n-1)r}{2}}}{(q;q)_n (b_1:q)_n (b_2;q)_n \ldots (b_{p+r};q)_n} $$

The series ${}_{p+1}\phi_{p+r}$ converges for all positive integers r and for all x. For $r = 0$,it converges only when $|x|<1$.

Jacobi's Triple Product Identity: (See [3], (2.2.10) and (2.2.11))

$(zq^{1/2}, z^{-1}q^{1/2}, q; q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n z^n q^{\frac{n^2}{2}}$, and its corollary is given by

$$ \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{(2k+1)n(n+1)}{2}-in} = \sum_{n=0}^{\infty}(-1)^n q^{(2k+1)n(n+1)-in}.(1-q^{(2n+1)i}) $$
$$ = \prod_{n=0}^{\infty}(1-q^{(2k+1)(n+1)})(1-q^{(2k+1)n+i})(1-q^{(2k+1)(n+1)-i}) \tag{1.1} $$

The following Lemma is due to W. N. Bailey: (See [1], (2.1.5))
If p is a non-negative integer, then

$$ (aq;q)_\infty \sum_{n=0}^{\infty} a^n.q^{n^2-pn}.\beta_n $$

$$ = \sum_{j=0}^{p}\frac{(q^{-p};q)_j(-a)^j q^{j(j+1)/2}}{(q;q)_j}.\sum_{n=0}^{\infty} a^n.q^{n^2-pn+2nj}.\alpha_n \tag{1.2} $$

Where

$$ \beta_n = \sum_{k=0}^{n}\frac{\alpha_k}{(q;q)_{n-k}(aq;q)_{n+k}}. $$

In this section, we begin by introducing the following transformations:

$$ {}_{10}\phi_9\left(\begin{matrix} a,q\sqrt{a},-q\sqrt{a},b,x,-x,y,-y,q^{-n},-q^{-n};q;-\dfrac{a^3q^{3+2n}}{bx^2y^2} \\ \sqrt{a},-\sqrt{a},\dfrac{aq}{b},\dfrac{aq}{x},-\dfrac{aq}{x},\dfrac{aq}{y},-\dfrac{aq}{y},-aq^{n+1},aq^{n+1} \end{matrix}\right) $$

$$ = \frac{(a^2q^2;q^2)_n\left(\dfrac{a^2q^2}{x^2y^2};q^2\right)_n}{\left(\dfrac{a^2q^2}{x^2};q^2\right)_n\left(\dfrac{a^2q^2}{y^2};q^2\right)_n}\cdot{}_5\phi_4\left(\begin{matrix} x^2,y^2,-\dfrac{aq}{b},-\dfrac{aq^2}{b},q^{-2n};q^2;q^2 \\ -aq,-aq^2,\dfrac{a^2q^2}{b^2},\dfrac{x^2y^2}{a^2}q^{-2n} \end{matrix}\right) \tag{1.3} $$

$$ {}_{12}\phi_{11}\left(\begin{matrix} a,q^3\sqrt{a},-q^3\sqrt{a},x,xq,xq^2,y,yq,yq^2,q^{-n},q^{-n+1},q^{-n+2};q^3;\dfrac{a^4q^{3+3n}}{x^3y^3} \\ \sqrt{a},-\sqrt{a},\dfrac{aq^3}{x},\dfrac{aq^2}{x},\dfrac{aq}{x},\dfrac{aq^3}{y},\dfrac{aq^2}{y},\dfrac{aq}{y},aq^{3+n},aq^{2+n},aq^{1+n} \end{matrix}\right) $$

$$ = \frac{(aq;q)_n\left(\dfrac{aq}{xy};q\right)_n}{\left(\dfrac{aq}{x};q\right)_n\left(\dfrac{aq}{y};q\right)_n}\cdot{}_6\phi_5\left(\begin{matrix} a^{\frac{1}{3}},\omega a^{\frac{1}{3}},\omega^2 a^{\frac{1}{3}},x,y,q^{-n};q;q \\ a^{\frac{1}{2}},-a^{\frac{1}{2}},(aq)^{\frac{1}{2}},-(aq)^{\frac{1}{2}},\dfrac{xy}{a}q^{-n} \end{matrix}\right) \tag{1.4} $$

Proof of (1.3)-(1.4): (See [1], Equation (1.3) and (1.6) respectively).

Multiple Series Generalization of Transformation (1.3): (See [1], (4.1)).

By induction on p, the multiple series generalization of (1.3) can be given in the form:

$$ {}_{2p+4}\phi_{2p+3}\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, x, -x, y, -y, (c_{p-3}), (d_{p-3}), -q^{-n}, q^{-n}; q; \dfrac{-a^p q^{p+2n}}{bx^2y^2c_1d_1 \ldots c_{p-3}d_{p-3}} \\ \sqrt{a}, -\sqrt{a}, \dfrac{aq}{b}, \dfrac{aq}{x}, -\dfrac{aq}{x}, \dfrac{aq}{y}, -\dfrac{aq}{y}, \dfrac{aq}{(c_{p-3})}, \dfrac{aq}{(d_{p-3})}, -aq^{n+1}, aq^{n+1} \end{matrix}\right) $$

$$ = \frac{(a^2q^2;q^2)_n \left(\dfrac{a^2q^2}{x^2y^2};q^2\right)_n}{\left(\dfrac{a^2q^2}{x^2};q^2\right)_n \left(\dfrac{a^2q^2}{y^2};q^2\right)_n} . \sum_{r_1, r_2, \ldots r_{p-3}} \prod_{j=1}^{p-3} \left\{ \frac{\left(\dfrac{aq}{c_jd_j};q\right)_{r_j} (c_j;q)_{M_{j-1}} (d_j;q)_{M_{j-1}}}{(q;q)_{r_j} \left(\dfrac{aq}{c_j};q\right)_{M_j} \left(\dfrac{aq}{d_j};q\right)_{M_j}} \right\} $$

$$. \frac{(b;q)_{M_{p-3}} (x^2;q^2)_{M_{p-3}} (y^2;q^2)_{M_{p-3}} (q^{-2n};q^2)_{M_{p-3}} q^{M_{p-3}(M_{p-3}+1)/2}}{(-aq;q)_{2M_{p-3}} \left(\dfrac{aq}{b};q\right)_{M_{p-3}} \left(\dfrac{x^2y^2}{a^2} q^{-2n}; q^2\right)_{M_{p-3}}}. $$

$$ \times \left(-\frac{a}{b}q^2\right)^{r_{p-3}} . \left(-\frac{a^2q^3}{bc_{p-3}d_{p-3}}q^2\right)^{r_{p-4}} \ldots \left(-\frac{a^{p-3}q^{p-2}}{bc_{p-3}.d_{p-3} \ldots c_2d_2}q^2\right)^{r_1} $$

$$ {}_5\phi_4\left(\begin{matrix} x^2q^{2M_{p-3}}, y^2q^{2M_{p-3}}, \dfrac{-aq^{1+M_{p-3}}}{b}, \dfrac{-aq^{2+M_{p-3}}}{b}, q^{-2n+2M_{p-3}}; q^2; q^2 \\ -aq^{1+2M_{p-3}}, -aq^{2+2M_{p-3}}, \dfrac{a^2}{b^2}q^{2+2M_{p-3}}, \dfrac{x^2y^2}{a^2}q^{-2n+2M_{p-3}} \end{matrix}\right) \tag{1.5} $$

Where $M_i = r_1 + r_2 + \cdots + r_i$, $M_{-1} = M_0 = 0$ and $(a_{M,N})$ stands for the $(N - M + 1)$ symbols $a_M, a_{M+1}, \ldots, a_N$ (when $M = 1$, we write (a_N) in place of $(a_{1,N})$.

Similarly the multiple series generalization of (1.4) can be given in the form: **(See [1], (4.5)).**

$$ {}_{2p+4}\phi_{2p+3}\left(\begin{matrix} a, q^3\sqrt{a}, -q^3\sqrt{a}, x, xq, xq^2, y, yq, yq^2, (c_{p-4}), (d_{p-4}), q^{-n}, q^{-n+1}, q^{-n+2}; q^3; \dfrac{a^p q^{3p-9+3n}}{x^3y^3c_1d_1 \ldots c_{p-4}d_{p-4}} \\ \sqrt{a}, -\sqrt{a}, \dfrac{aq^3}{x}, \dfrac{aq^2}{x}, \dfrac{aq}{x}, \dfrac{aq^3}{y}, \dfrac{aq^2}{y}, \dfrac{aq}{y}, \dfrac{aq^3}{(c_{p-4})}, \dfrac{aq^3}{(d_{p-4})}, aq^{3+n}, aq^{2+n}, aq^{1+n} \end{matrix}\right) = $$

$$ \frac{(aq;q)_n \left(\dfrac{aq}{xy};q\right)_n}{\left(\dfrac{aq}{x};q\right)_n \left(\dfrac{aq}{y};q\right)_n} \sum_{r_1, r_2, \ldots r_{p-4}} \prod_{j=1}^{p-4} \left\{ \frac{\left(\dfrac{aq^3}{c_jd_j};q^3\right)_{r_j} (c_j;q^3)_{M_{j-1}} (d_j;q^3)_{M_{j-1}}}{(q^3;q^3)_{r_j} \left(\dfrac{aq^3}{c_j};q^3\right)_{M_j} \left(\dfrac{aq^3}{d_j};q^3\right)_{M_j}} \right\}. $$

$$. \frac{(x;q)_{3M_{p-4}} (y;q)_{3M_{p-4}} (q^{-n};q)_{3M_{p-4}} (aq^3;q^3)_{2M_{p-4}} q^{3M_{p-4}(M_{p-4}+1)}}{(aq;q)_{6M_{p-4}} \left(\dfrac{xy}{a}q^{-n};q\right)_{3M_{p-4}}}. $$

$$ (a)^{r_{p-4}} \left(\frac{a^2q^3}{c_{p-4}d_{p-4}}\right)^{r_{p-5}} \ldots \left(\frac{a^{p-4}q^{3p-15}}{c_{p-4}.d_{p-4} \ldots c_2d_2}q^2\right)^{r_1}. $$

$$ {}_6\phi_5\left(\begin{matrix} a^{\frac{1}{3}}q^{2M_{p-4}}, \omega a^{\frac{1}{3}}q^{2M_{p-4}}, \omega^2 a^{\frac{1}{3}}q^{2M_{p-4}}, xq^{3M_{p-4}}, yq^{3M_{p-4}}, q^{-n+3M_{p-4}}; q; q \\ a^{\frac{1}{2}}q^{3M_{p-4}}, -a^{\frac{1}{2}}q^{3M_{p-4}}, a^{\frac{1}{2}}q^{\frac{1}{2}+3M_{p-4}}, -a^{\frac{1}{2}}q^{\frac{1}{2}+3M_{p-4}}, \dfrac{xy}{a}q^{-n+3M_{p-4}} \end{matrix}\right) \tag{1.6} $$

4for$p \geq 4$ (ω is the imaginary cube root of unity), and $M_i = r_1 + r_2 + \cdots + r_i$, $M_{-1} = M_0 = 0$.

Main Results

2. Rogers-Ramanujan Type Identities Modulo 15s:

Replacing $\boldsymbol{y}$ by $\boldsymbol{iq^{-n}}$and$\boldsymbol{q}$by $\boldsymbol{q^s}$in (1.3) and then letting $b, x \to \infty$ we find,

$$\sum_{k=0}^{n} \frac{(-1)^k (aq^s;q^s)_{k-1}(1-q^{2ks})a^{3k} q^{\frac{7k^2s-ks}{2}}}{(q^s;q^s)_k (a^4q^{4s};q^{4s})_{n+k}(q^{4s};q^{4s})_{n-k}}$$

$$= \frac{1}{(-a^2q^{2s};q^{2s})_{2n}} \sum_{k=0}^{n} \frac{a^{2k}q^{2k^2s}}{(q^{2s};q^{2s})_k(-aq^s;q^s)_{2k}(q^{4s};q^{4s})_{n-k}} \tag{2.1}$$

The left hand side of Bailey's Lemma (1.2) for $a = a^4$, $q = q^{4s}$, $\alpha_0 = 1$, $\alpha_{k+1} = 0$ and,

$\alpha_k = \frac{(-1)^k (aq^s;q^s)_{k-1}(1-q^{2ks})a^{3k} q^{\frac{7k^2s-ks}{2}}}{(q^s;q^s)_k}$, gives,

$$(a^2q^{2s};q^{2s})_\infty \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{a^{4n+6k}q^{4n^2s+6k^2s+8nks-4p(n+k)s}}{(q^{2s};q^{2s})_k(-aq^s;q^s)_{2k}(q^{4s};q^{4s})_n}, \text{ (upon using (2.1))} \tag{2.2}$$

The corresponding Right hand side of Bailey's Lemma (1.2) for the same, yields

$$\sum_{j=0}^{p} \frac{(q^{-4ps};q^{4s})_j (-1)^j a^{4j} q^{2j(j+1)s}}{(q^{4s};q^{4s})_j} . \sum_{n=0}^{\infty} a^{4n} . q^{(4n^2-4pn+8nj)s} . \frac{(-1)^n a^{3n}(aq^s;q^s)_{n-1}(1-q^{2ns})q^{\frac{7n^2s-ns}{2}}}{(q^s;q^s)_n} \tag{2.3}$$

We equate (2.2) and (2.3) to get,

$$(a^2q^{2s};q^{2s})_\infty \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{a^{4n+6k}q^{4n^2s+6k^2s+8nks-4p(n+k)s}}{(q^{2s};q^{2s})_k(-aq^s;q^s)_{2k}(q^{4s};q^{4s})_n}$$

$$\sum_{j=0}^{p} \frac{(q^{-4ps};q^{4s})_j (-1)^j a^{4j} q^{2j(j+1)s}}{(q^{4s};q^{4s})_j} . \sum_{n=0}^{\infty} \frac{(-1)^n a^{7n}(aq^s;q^s)_{n-1}(1-q^{2ns})q^{\frac{15n^2s-ns}{2}-4pns+8njs}}{(q^s;q^s)_n} \tag{2.4}$$

Setting $a = 1$ and then placing $p = 0, 1, 2$ successively in (2.4), we get the following Identities upon using (1.1):

$$(-q^s;q^s)_\infty \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{q^{4n^2s+6k^2s+8nks}}{(q^{2s};q^{2s})_k(-q^s;q^s)_{2k}(q^{4s};q^{4s})_n}$$

$$= \frac{1}{(q^s;q^s)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{15n^2s-ns}{2}}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1-q^n}, \text{ where } n \not\equiv 0, \pm 7s \ (\text{mod } 15s) \tag{2.5}$$

$$(-q^s;q^s)_\infty \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{q^{4n^2s+6k^2s+8nks-4ns-4ks}}{(q^{2s};q^{2s})_k(-q^s;q^s)_{2k}(q^{4s};q^{4s})_n}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1-q^n} + \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

Where $n \not\equiv 0, \pm 3$s (mod 15s) and $n \not\equiv 0, \pm 4$s (mod 15s) respectively. (2.6)

and,

$$(-q^s;q^s)_\infty \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{q^{4n^2s+6k^2s+8nks-8(n+k)s}}{(q^{2s};q^{2s})_k(-q^s;q^s)_{2k}(q^{4s};q^{4s})_n}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1-q^n} + \frac{1+q^{4s}}{q^{4s}} \prod_{n=1}^{\infty} \frac{1}{1-q^n} + \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

Where $n \not\equiv 0, \pm 1$s (mod 15s), $n \not\equiv 0, \pm 7$s (mod 15s) and $n \not\equiv 0$ (mod 15s) respectively. (2.7)

3. Rogers-Ramanujan Type Identities Modulo 9s: (where s is any finite positive integer)

Taking$q = q^s, b \to 0, x \to \infty, y \to \infty$and $a = a^2$in the transformation (1.3), we find,

$$\sum_{k=0}^{n} \frac{(-1)^k a^{4k}(a^2q^s;q^s)_{k-1}(1-a^2q^{2ks})q^{(5k^2s-ks)/2}}{(q^s;q^s)_k(q^{2s};q^{2s})_{n-k}(a^4q^{2s};q^{2s})_{n+k}} = \sum_{k=0}^{n} \frac{(-1)^k a^{4k}q^{3k^2s}}{(q^{2s};q^{2s})_k(-q^s;q^s)_{2k}(q^{2s};q^{2s})_{n-k}} \tag{3.1}$$

The Bailey's Lemma (1.2), for $a = a^4$, $q = q^{2s}$, $\alpha_0 = 1$, $\alpha_{k+1} = 0$and,

$\alpha_k = \frac{(-1)^k a^{4k}(a^2q^s;q^s)_{k-1}(1-a^2q^{2ks})q^{(5k^2s-ks)/2}}{(q^s;q^s)_k}$, gives (upon using (3.1)),

$$(a^4q^{2s};q^{2s})_\infty \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k a^{4n+8k}q^{2n^2s+5k^2s+4nks-2p(n+k)s}}{(q^{2s};q^{2s})_k(-q^s;q^s)_{2k}(q^{2s};q^{2s})_n}$$

$$=\sum_{j=0}^{p}\frac{(q^{-2ps};q^{2s})_j\,(-1)^j a^{4j}q^{j(j+1)s}}{(q^{2s};q^{2s})_j}.\sum_{n=0}^{\infty}\frac{(-1)^n a^{8n}(a^2q^s;q^s)_{n-1}(1-a^2q^{2ns})q^{\frac{9n^2s-4pns+8njs-ns}{2}}}{(q^s;q^s)_n} \tag{3.2}$$

Setting $a = 1$ and then placing $p = 0, 1, 2$ successively in (3.2), we get the following Rogers-Ramanujan Type Identities after using the Jacobi's Triple Product Identity (1.1):

i). $$(-q^s;q^s)_\infty \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{2n^2s+5k^2s+4nks}}{(q^{2s};q^{2s})_k(-q^s;q^s)_{2k}(q^{2s};q^{2s})_n}$$

$$=\frac{1}{(q^s;q^s)_\infty}\sum_{n=0}^{\infty}(-1)^n(1+q^{ns})q^{(9n^2s-ns)/2}$$

$$=\frac{1}{(q^s;q^s)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{(9n^2s+ns)/2}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n},\ \text{where } n \not\equiv 0, \pm 4s \pmod{9s} \tag{3.3}$$

ii). $$(-q^s;q^s)_\infty \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{2n^2s+5k^2s+4nks-2ns-2ks}}{(q^{2s};q^{2s})_k(-q^s;q^s)_{2k}(q^{2s};q^{2s})_n}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n}+\prod_{n=1}^{\infty}\frac{1}{1-q^n} \tag{3.4}$$

Where $n \not\equiv 0, \pm$ 2s (mod 9s) and $n \not\equiv 0, \pm$ 3s (mod 9s) respectively.

iii). $$(-q^s;q^s)_\infty \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{2n^2s+5k^2s+4nks-4ns-4ks}}{(q^{2s};q^{2s})_k(-q^s;q^s)_{2k}(q^{2s};q^{2s})_n}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n}+\frac{1+q^{2s}}{q^{2s}}\prod_{n=1}^{\infty}\frac{1}{1-q^n}+\prod_{n=1}^{\infty}\frac{1}{1-q^n} \tag{3.5}$$

Where $n \not\equiv 0, \pm$ 1s (mod 9s), $n \not\equiv 0, \pm$ 4s (mod 9s) and $n \not\equiv 0, \pm$ 0 (mod 9s) respectively.

For $s = 1$, taking $a = q^{\frac{1}{2}}$, $p = 0$ in (3.2), we have the following:

$$(-q^2;q)_\infty \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{2n^2+5k^2+4nk+2n+4k}}{(q^2;q^2)_k(-q;q)_{2k}(q^2;q^2)_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n},\ n \not\equiv 0, \pm 1 \pmod 9 \tag{3.6}$$

Also, replacing y by $x^{1/2}q^{-n/2}$, q by q^s in (1.3) and then letting $b \to 0, x \to \infty$, we find that,

$$\sum_{k=0}^{n}\frac{(-1)^k(aq^s;q^s)_{k-1}(1-aq^{2ks})a^{2k}q^{\frac{5k^2s-ks}{2}}}{(q^s;q^s)_k(a^2q^{2s};q^{2s})_{n+k}(q^{2s};q^{2s})_{n-k}}=\sum_{k=0}^{n}\frac{(-1)^k a^{2k}q^{2k^2-k}}{(q^{2s};q^{2s})_k(-aq^s;q^s)_{2k}\,(q^{2s};q^{2s})_{n-k}} \tag{3.7}$$

The Bailey's Lemma (1.2) for $a = a^2$, $q = q^{2s}$, $\alpha_0 = 1$, $\alpha_{k+1} = 0$ and,

$\alpha_k = \frac{(-1)^k(aq^s;q^s)_{k-1}(1-aq^{2ks})a^{2k}q^{\frac{5k^2s-ks}{2}}}{(q^s;q^s)_k}$, gives (upon using (3.7)),

$$(a^2q^{2s};q^{2s})_\infty \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k a^{2n+4k}q^{2n^2s+4k^2s+4nks-2pns-2pks-ks}}{(q^{2s};q^{2s})_k(-aq^s;q^s)_{2k}(q^{2s};q^{2s})_n}$$

$$=\sum_{j=0}^{p}\frac{(q^{-2ps};q^{2s})_j\,(-1)^j a^{2j}q^{j(j+1)s}}{(q^{2s};q^{2s})_j}.\sum_{n=0}^{\infty}\frac{(-1)^n a^{4n}.(aq^s;q^s)_{n-1}(1-aq^{2ns})q^{\frac{(9n^2s-4pns+8njs-ns)}{2}}}{(q^s;q^s)_n} \tag{3.8}$$

Setting $a = 1$ and then placing $p = 0, 1, 2$ successively in (3.8), we get the following Identities:

$$(-q^s;q^s)_\infty \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{2n^2s+4k^2s+4nks-ks}}{(q^{2s};q^{2s})_k(-q^s;q^s)_{2k}(q^{2s};q^{2s})_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n},$$

Where $n\not\equiv 0,\pm 4s \pmod{9s}$ (3.9)

$$(-q^s;q^s)_\infty \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{2n^2s+4k^2s+4nks-2ns-3ks}}{(q^{2s};q^{2s})_k(-q^s;q^s)_{2k}(q^{2s};q^{2s})_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n}+\prod_{n=1}^{\infty}\frac{1}{1-q^n} \tag{3.10}$$

Where $n\not\equiv 0,\pm 2s \pmod{9s}$ and $n\not\equiv 0,\pm 3s \pmod{9s}$ respectively.

and,

$$(-q^s;q^s)_\infty \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{2n^2s+4k^2s+4nks-4ns-5ks}}{(q^{2s};q^{2s})_k(-q^s;q^s)_{2k}(q^{2s};q^{2s})_n}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n}+\frac{1+q^2}{q^2}\prod_{n=1}^{\infty}\frac{1}{1-q^n}+\prod_{n=1}^{\infty}\frac{1}{1-q^n}$$

Where $n\not\equiv 0 \pmod{9s}$, $n\not\equiv 0,\pm 4s \pmod{9s}$ and $n\not\equiv 0,\pm 8s \pmod{9s}$ respectively. (3.11)

4. Rogers-Ramanujan Type Identities Modulo $11s$:

Setting $p=4, c_1=z, d_1=-z, q=q^s$ and then taking $b\to 0, x,y,z\to\infty$ in the transformation (1.5), we get after some simplification,

$$\sum_{k=0}^{n}\frac{(-1)^k(aq^s;q^s)_{k-1}(1-aq^{2ks})a^{3k}\,q^{(7k^2s-ks)/2}}{(q^s;q^s)_k(q^{2s};q^{2s})_{n-k}(a^2q^{2s};q^{2s})_{n+k}}=\sum_{r=0}^{\infty}\sum_{k=0}^{n-r}\frac{(-1)^k a^{2k+2r}q^{5r^2s+8rks+2k^2s-2ks+rs}}{(q^s;q^s)_r(-aq^s;q^s)_{2r+2k}(q^{2s};q^{2s})_{n-r-k}} \tag{4.1}$$

The Bailey's Lemma (1.2) for $a=a^2$, $q=q^{2s}$, $\alpha_0=1$, $\alpha_{k+1}=0$ and,

$\alpha_k=\dfrac{(-1)^k(aq^s;q^s)_{k-1}(1-aq^{2ks})a^{3k}\,q^{(7k^2s-ks)/2}}{(q^s;q^s)_k}$ gives (upon using (4.1)),

$$(a^2q^{2s};q^{2s})_\infty \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k a^{2n+4k}\,q^{(2n^2+7r^2+4k^2-4nr+4rk-2k+r)s-2ps(n-r+k)}}{(q^s;q^s)_r(-aq^s;q^s)_{2r+2k}(q^{2s};q^{2s})_{n-2r}}$$

$$=\sum_{j=0}^{p}\frac{(q^{-2ps};q^{2s})_j\,(-1)^j a^{2j}q^{j(j+1)s}}{(q^{2s};q^{2s})_j}.\sum_{n=0}^{\infty}\frac{(-1)^n(aq^s;q^s)_{n-1}(1-aq^{2ns})a^{5n}\,q^{(11n^2s-4pns+8njs-ns)/2}}{(q^s;q^s)_n} \tag{4.2}$$

Setting $a=1$ and then placing $p=0,1,2$ successively in (4.2), we get the following:

$$(-q^s;q^s)_\infty \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{(2n^2+7r^2+4k^2-4nr+4rk-2k+r)s}}{(q^s;q^s)_r(-q^s;q^s)_{2r+2k}(q^{2s};q^{2s})_{n-2r}}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n},\ \text{where } n\not\equiv 0,\pm 5s \pmod{11s} \tag{4.3}$$

$$(-q^s;q^s)_\infty \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{(2n^2+7r^2+4k^2-4nr+4rk-4k+3r-2n)s}}{(q^s;q^s)_r(-q^s;q^s)_{2r+2k}(q^{2s};q^{2s})_{n-2r}}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n}+\prod_{n=1}^{\infty}\frac{1}{1-q^n} \tag{4.5}$$

Where $n\not\equiv 0,\pm 3s \pmod{11s}$ and $n\not\equiv 0,\pm 4s \pmod{11s}$ respectively.

and,

$$(-q^s;q^s)_\infty \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{(2n^2+7r^2+4k^2-4nr+4rk-6k+5r-4n)s}}{(q^s;q^s)_r(-q^s;q^s)_{2r+2k}(q^{2s};q^{2s})_{n-2r}}$$

$$=\prod_{n=1}^{\infty}\frac{1}{1-q^n}+\frac{1+q^2}{q^2}\prod_{n=1}^{\infty}\frac{1}{1-q^n}+\prod_{n=1}^{\infty}\frac{1}{1-q^n}$$

Where $n\not\equiv 0,\pm 1 \pmod{11s}$, $n\not\equiv 0,\pm 5s \pmod{11s}$ and $n\not\equiv 0,\pm 2s \pmod{11s}$ respectively. (4.6)

5. Rogers-Ramanujan Type Identities modulo 45s: (where s is any finite positive integer)

Taking $q = q^s, x \to \infty, y \to \infty$ in the transformation (1.4), we find that,

$$\sum_{k=0}^{n} \frac{(-1)^k a^{4k} (aq^{3s};q^{3s})_{k-1}(1-aq^{6ks}) q^{\frac{27k^2s-3ks}{2}}}{(q^{3s};q^{3s})_k (q^s;q^s)_{n-3k}(aq^s;q^s)_{n+3k}} = \sum_{k=0}^{n} \frac{a^k (aq^{3s};q^{3s})_{k-1} q^{k^2 s}}{(q^s;q^s)_k (aq^s;q^s)_{2k-1}(q^s;q^s)_{n-k}} \tag{5.1}$$

The Bailey's Lemma (1.2) for $q = q^s$, $\alpha_0 = 1$, $\alpha_{3k+1} = 0$ and,

$\alpha_{3k} = \frac{(-1)^k a^{4k}(aq^{3s};q^{3s})_{k-1}(1-aq^{6ks})q^{\frac{27k^2s-3ks}{2}}}{(q^{3s};q^{3s})_k}$ gives (upon using (5.1)),

$$(aq^s;q^s)_\infty \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{a^{n+2k}(aq^{3s};q^{3s})_{k-1} q^{(n^2+2nk+2k^2)s-p(n+k)s}}{(q^s;q^s)_k (aq^s;q^s)_{2k-1}(q^s;q^s)_n}$$

$$= \sum_{j=0}^{p} \frac{(q^{-ps};q^s)_j (-1)^j a^j q^{\frac{j(j+1)s}{2}}}{(q^s;q^s)_j} . \sum_{n=0}^{\infty} \frac{(-1)^n a^{7n}(aq^{3s};q^{3s})_{n-1}(1-aq^{6ns}) q^{\frac{27n^2s-3ns}{2}+9n^2s-3pns+6njs}}{(q^{3s};q^{3s})_n} \tag{5.2}$$

Setting $a = 1$ and then placing $p = 0, 1, 2$ successively in (5.2), we get the following:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(q^{3s};q^{3s})_{k-1} q^{(n^2+2nk+2k^2)s}}{(q^s;q^s)_k (q^s;q^s)_{2k-1}(q^s;q^s)_n} = \prod_{n=1}^{\infty}\frac{1}{1-q^n}, \text{ where } n \not\equiv 0, \pm 24\text{s (mod 45s)} \tag{5.3}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(q^{3s};q^{3s})_{k-1} q^{(n^2+2nk+2k^2)s-ns-ks}}{(q^s;q^s)_k (q^s;q^s)_{2k-1}(q^s;q^s)_n} = \prod_{n=1}^{\infty}\frac{1}{1-q^n} + \prod_{n=1}^{\infty}\frac{1}{1-q^n} \tag{5.4}$$

Where $n \not\equiv 0, \pm$ 27s (mod 45s) and $n \not\equiv 0, \pm$ 24s (mod 45s) respectively.

and,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(q^{3s};q^{3s})_{k-1} q^{(n^2+2nk+2k^2)s-2ns-2ks}}{(q^s;q^s)_k (q^s;q^s)_{2k-1}(q^s;q^s)_n}$$

$$= \prod_{n=1}^{\infty}\frac{1}{1-q^n} + \frac{1+\mathrm{q}}{\mathrm{q}}\prod_{n=1}^{\infty}\frac{1}{1-q^n} + \prod_{n=1}^{\infty}\frac{1}{1-q^n} \tag{5.4}$$

Where $n \not\equiv 0, \pm 30$ (mod 45s), $n \not\equiv 0, \pm$ 27s (mod 45s) and $n \not\equiv 0, \pm 24s$ (mod 45s) respectively.

6. Some Particular Cases:

Taking $s = 1,2,3, \ldots$ …. successively in the identities (2.5) – (2.7), we get many identities of the Rogers-Ramanujan Type modulo 15 and integral multiples of 15 in succession. For instance, if we put $s = 1,2,3$, …. in (2.5), it gives the following identities:

$$(-q;q)_\infty \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{q^{4n^2+6k^2+8nk}}{(q^2;q^2)_k(-q;q)_{2k}(q^4;q^4)_n} = \prod_{n=1}^{\infty}\frac{1}{1-q^n}, \text{ where } n \not\equiv 0, \pm 7 \text{ (mod 15)} \tag{6.1}$$

$$(-q^2;q^2)_\infty \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{q^{8n^2+12k^2+16nk}}{(q^4;q^4)_k(-q^2;q^2)_{2k}(q^8;q^8)_n} = \prod_{n=1}^{\infty}\frac{1}{1-q^n},$$

Where $n \not\equiv 0, \pm$ 14 (mod 30) (6.2) $(-q^3;q^3)_\infty \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{q^{12n^2+18k^2+24nk}}{(q^6;q^6)_k(-q^3;q^3)_{2k}(q^{12};q^{12})_n} = \prod_{n=1}^{\infty}\frac{1}{1-q^n}$, where $n \not\equiv 0, \pm$ 21 (mod 45) (6.3) and so on.

For $s = 1,2,3$, …. in (2.6), it gives

$$(-q;q)_\infty \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{q^{4n^2+6k^2+8nk-4n-4k}}{(q^2;q^2)_k(-q;q)_{2k}(q^4;q^4)_n} = \prod_{n=1}^{\infty}\frac{1}{1-q^n} + \prod_{n=1}^{\infty}\frac{1}{1-q^n} \tag{6.4}$$

Where $n \not\equiv 0, \pm$ 3 (mod 15) and $n \not\equiv 0, \pm$ 4 (mod 15) respectively.

$$(-q^2;q^2)_\infty \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{q^{8n^2+12k^2+16nk-8n-8k}}{(q^4;q^4)_k(-q^2;q^2)_{2k}(q^8;q^8)_n} = \prod_{n=1}^{\infty}\frac{1}{1-q^n} + \prod_{n=1}^{\infty}\frac{1}{1-q^n} \tag{6.5}$$

Where $n \not\equiv 0, \pm$ 6 (mod 30) and $n \not\equiv 0, \pm$ 8 (mod 30) respectively.

and so on.

Similarly,varying s over 1, 2, 3, …… in (3.3) – (3.5) and (3.9) – (3.11), we get Rogers-Ramanujan Type identities modulo 9,18,27,……… onwards. Also, varying s over 1, 2, 3, …… onwards in (4.3) – (4.6) and (5.3) – (5.4) we get identities modulo 11, 22, 33,……… and 45, 90, 135, ……..onwards respectively.

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