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Pool Maximum Violation Region Algorithm to Obtain Closest Nonincreasing Function under Various Norms

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Abstract

We came across estimation of an unknown probability density function. This unknown density function itself may have some known constraints. Therefore, it is trivial to expect the same constraints from its density estimator too. In this article we consider density estimator is given or already estimated but it may not satisfy requirement of non-increasingness property from its known modal point. Without loss of generality we assume mode is equal to zero. Hence, we provide an algorithm to obtain closest non-increasing function to a given density estimator under given measures of closeness.

We have described Pool Maximum Violation Region Algorithm (PMVRA) for obtaining closest non-increasing function to a given one. First we obtain maximum violation region of a given function, if violation of non-increasingness exists. After doing so, we give a closest constant value on the superset of maximum violation region when measure of closeness is Sup-norm, L_1-norm and L_2-norm.

In case of Sup-norm, we found closest constant on the superset of maximum violation region as arithmetic mean of smallest and largest values of the function on this superset. When measure of closeness is L_1-norm, we found closest constant on the superset of maximum violation region as median of the function on this superset. Whereas, in case of measure of closeness is L_2-norm, we found closest constant on the superset of maximum violation region as an arithmetic mean of the function on this superset.

Keywords: Non-increasingness, maximum violation region, measure of closeness, sup-norm, l_1-norm, l_2-norm, pool maximum violation region algorithm (PMVRA)

1. Introduction

Many times we came across estimation of density function (Silverman B.W. (1986) [7]. Though density function is unknown, it may have some known restrictions on it, like symmetry or unimodality or decreasing nature as its domain goes away from its modal value etc. (Chaubey Y.P. et. al. (2012) [1], Lo S. H. (1985) [2], Schuster E.G. (1975) [75]. Therefore, these are trivial expected constraints on its estimator too.

When there is a restriction of decreasing or non-increasingness nature on a density estimator on its positive support, we have to modify it in such a way that it becomes non-increasing as well as closest to given density estimator.

This paper contributes Pool Maximum Violation Region Algorithm (PMVRA). First it finds maximum violation region [u, v] and then removes violation property of a function over region [u, v] along with rectification of function over a superset D of [u, v] by a constant value L^*. This L^* is close to the function over a superset D of [u, v] and closeness is measured on the basis of given norm.

In every iteration of PMVRA, function is rectified over superset D of [u, v], therefore this algorithm requires minimum number of iterations to obtain closest and non-increasing function to a given one.

In next section we propose Pool Maximum Violation Region Algorithm and then obtain closest non-increasing function to a given density estimator closest under Sup-norm, L_1-norm and L_2-norm (Randles R. and Wolfe D. (1979) [3], Robertson T., Wright F.T. and Dykstra R.L.(1988) [4], Schuster E.G. (1991) [6].

2. Pool Maximum Violation Region Algorithm (PMVRA)

Let f(x): $R^+ \rightarrow R^+$ be an arbitrary function. In the following, for notational convenience we refer the *non-increasing property of a function* as P. Let V (f) = {x: f(x) is strictly increasing at x}, the set on which f(.) violates P. An interval (u, v) is said to be violation interval of f(.) if the function f(.) does not satisfy P on the interval (u, v). An interval (a, b) is said to be a *maximal violation interval* of the function f(.) if f(b)-f(a) ≥ f(b`) - f(a`), for any other violation interval (a`, b`). We modify the function f(.) to a function f_1(.) by modifying the function f(.) by a suitable constant L on maximal violation interval of f(.). However, such constants at each stage of modification can be selected optimally under a specific norm (distance function). We consider the norms: (a) Sup- norm (b) L_1- norm and (c) L_2- norm. If f(.) does not violates P then there is no need to modify f(x). If f(.) violates the property P,

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then there exists at least one violation interval. We assume that f(.) satisfies the following conditions:

i). f(x) is finite.

ii). $\int_0^\infty f(x)dx < \infty$, ($\int_0^\infty f(x)dx = 1$, (without loss of generality).

iii). f(x) has a finite number of turning points (points of local maxima or minima).

2.1. Development of the Algorithm

Following is an algorithm to modify an arbitrary function f(.) to a non-increasing function. This modified function is piecewise constant on the set D, where D is the set on which f(.) is being modified.

Step 1: *Select the function f(.).*

Step 2: *Test the given function f(.) for its violation of P.*

If there is no violation of P (that is, V (f) = Φ) then stop. Else go to Step-3.

Step-3: *Determination of modified function*:

Let (a, b) be a maximal violation interval of the function f(.).

For L, $f(a) \leq L \leq f(b)$,

let $a_1(L) = \inf\{x : f(x) \leq L\}$, $b_1(L) = \sup\{x : f(x) \geq L\}$,

$A(L) = \{x : a_1(L) \leq x < \infty, L \leq f(x) < \infty\}$, $B(L) = \{x : 0 \leq x \leq b_1(L), 0 \leq f(x) \leq L\}$

$D(L) = A(L) \cup B(L) = [a_1(L), b_1(L)]$.

In the following, for notational simplicity we write $a_1(L)$, $b_1(L)$, A(L), B(L) and D(L) as a_1, b_1, A, B and D respectively. Note that, $a_1 \leq a < b \leq b_1$.

Define:

$$f_1(x, L) = \begin{cases} L, & \text{if } x \in D \\ f(x), & \text{if otherwise} \end{cases} \tag{1}$$

In the above, $f_1(x, L)$ (= $f_1(x)$ say) is the modified function and D is the domain for modification. It is to be noted that, $f_1(x) > L$ for $x \leq a_1$ and $f_1(x) < L$ for $x > b_1$. A typical function f(x) and the corresponding sets A(L) and B(L), for some L, (f(a) < L < f(b)) are described in Figure-1.

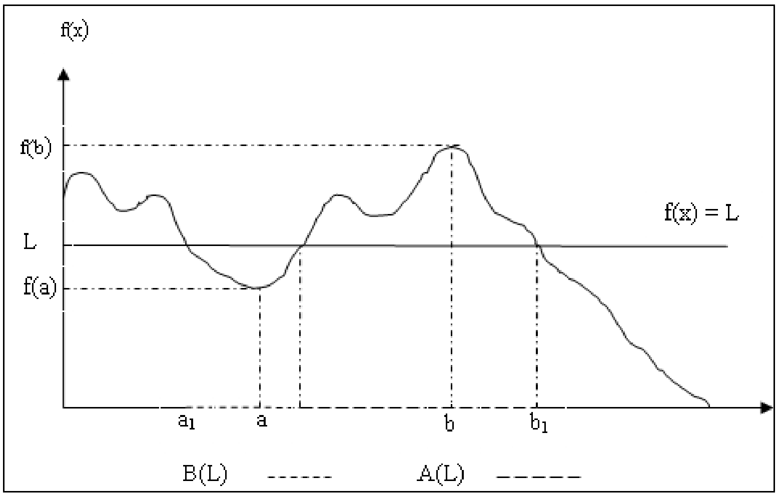

Fig 1: A Typical function with A(L) and B(L) for arbitrary L

Here, the interval (a, b) is *maximal violation interval* of the function f(.).

Step-4: *Identification of two functions on disjoint intervals (if exist)*:

If V (f_1) = Φ then declare that $f_1(.)$ is non-increasing and stop, else identify the two functions on the disjoint intervals (0, a_1) and (b_1,∞) given by:

$f_{11}(x) = f_1(x)$ for $0 < x < a_1$ and $f_{12}(x) = f_1(x)$ for $x > b_1$.

Stop identifying the domain for modification if V (f_{11}) ∪ V (f_{12}) = Φ, else go to Step-1 and replace f(.) by $f_{11}(.)$ and/ or $f_{12}(.)$ as the case may be.

In the above, the choice of constants L's is not unique, however these constants at each stage can be selected optimally so that the resulting modified function $f_1(x)$ is closest to the original function f(.) under a given distance function d(.). The algorithm described above is referred as the *Piecewise Maximum Violation Region Algorithm for the distance function d(.)* (PMVRA-d). In the following, we describe methods to obtain L^* for a given distance measures.

We note that the sets A(L) are decreasing in L, decrease from A(f(a)) to A(f(b)) = Φ, whereas B(L) are increasing in L, increase from B(f(a)) = Φ to B(f(b)). Furthermore, the difference (μ{A(L)}-μ{B(L)}) is decreasing and it has one change of sign at L (= L^* , say), where μ(.) is an appropriate measure of a set; for example the Lebesgue measure.

Corresponding to the maximal violation interval (a, b) we find L^*, ($f(a) < L^* < f(b)$) such that $f_l(x, L^*)$ is closest (under a norm) to the function f(.). We note that, $f_l(.)$ depends on L and the given function f(.); and hence for given f(.), d(f, f_l), the distance between f(.) and $f_l(.)$ depends only on L. Let

$\delta(L) = d(f, f_l)$, for $f(a) \le L \le f(b)$. (2)

Hence, to find the closest function $f_l(.)$ to f(.), it is enough to find L^* such that

$\delta(L^*) = \text{Infimum}\{\delta(L); L \in [f(a), f(b)]\}$ (3)

Depending upon choices of distance measures d(.), we obtain respective L^* 's and develop the respective PMVRA-d.

As $f_l(x, L) = f(x)$ for $x \notin D$, we have, $d(f, f_l) = d(f^D, f_1^D)$, where f^D (f_1^D) is the confined function defined on the domain D obtained from f (f_l). Further, as $D = A \cup B$ and $A \cap B = \Phi$, we have

$$d_S(f^D, f_1^D) = Maximum\{d_S(f^A, f_1^A), d_S(f^B, f_1^B)\} \tag{4}$$

$$d_j(f^D, f_1^D) = d_j(f^A, f_1^A) + d_j(f^B, f_1^B) \text{ for } j = 1, 2 \tag{5}$$

where $d_S(.)$ and $d_j(.)$ are sup-norm and L_j -norm, respectively. In the following we state and prove Lemma-1 and Lemma-2 and these are used in PMVRA under L_1-norm and L_2-norm. To prove these lemmas we assume that: $f : f(x) : R^+ \to R^+$ and $\int_0^\infty f(x)dx < \infty$. Let L is an arbitrary constant such that Infimum{f(x); x ∈ R} < L < Suprimum{f(x); x ∈ R} and A(L) = {x : f(x) > L}.

Lemma-1: If $I_1(L) = \int_{A(L)} (f(x) - L)dx$ then

$$\frac{d}{dL} I_1(L) = -\Lambda\{A(L)\} \tag{6}$$

Proof: Rewrite $I_1(L)$ as, $I_1(L) = \int_{A(L)} \left(\int_0^{(f(x)-L)} dy\right) dx = \Lambda^2(S_L)$, where $S_L = \{(x, y) : x \in A(L), L < y < f(x)\}$ and $\Lambda^2(.)$ is an area measure on a set. By definition of S_L, note that $S_{L+\epsilon} \subset S_L$ for any $\epsilon > 0$. Now consider, $I_1(L) - I_1(L+\epsilon)$. See Figure 2.

$$\begin{aligned}
& I_1(L) - I_1(L+\epsilon) \\
&= \Lambda^2(S_{L+\epsilon}) - \Lambda^2(S_L) \\
&= -[\Lambda^2(S_L) - \Lambda^2(S_{L+\epsilon})] \\
&= -[\Lambda^2(S_L - S_{L+\epsilon})], \because S_{L+\epsilon} \subset S_L \\
&= -\Lambda^2(W_{(L,\epsilon)}), \text{ where } W_{(L,\epsilon)} = S_L - S_{L+\epsilon} \\
&= -\Lambda^2[(u, v): 0 < v < \epsilon,\ u \in W^v_{(L,\epsilon)}], \text{ where } W^v_{(L,\epsilon)} = \{(u, v) \in W_{(L,\epsilon)}\}: && \text{(: v-section of set } W_{(L,\epsilon)}). \\
&= -\int_0^\epsilon \left(\int_{W^v_{(L,\epsilon)}} du\right) dv = -\int_0^\epsilon g(v, \epsilon)dv, \text{ where } g(v, \epsilon) = \int_{W^v_{(L,\epsilon)}} du: && \text{(a function of (v, u) only.}
\end{aligned}$$

$$\therefore lim_{\epsilon \to 0} \frac{I_1(L+\epsilon) - I_1(L)}{\epsilon} = lim_{\epsilon \to 0} - \int_0^\epsilon g(v, \epsilon)dv = -g(0, 0) = \int_{W^0_{(L,0)}} du = -\Lambda\{A(L)\},$$

Where $\Lambda(.)$ is a Lebesgue measure of a set. Hence we have proved that,

$$\frac{d}{dL} I_1(L) = lim_{\epsilon \to 0} \frac{I_1(L+\epsilon) - I_1(L)}{\epsilon} = -\Lambda\{A(L)\}.$$

Remark: If A(L) = {x : f(x) < L} then $\frac{d}{dL} I_1(L) = \Lambda\{A(L)\}$

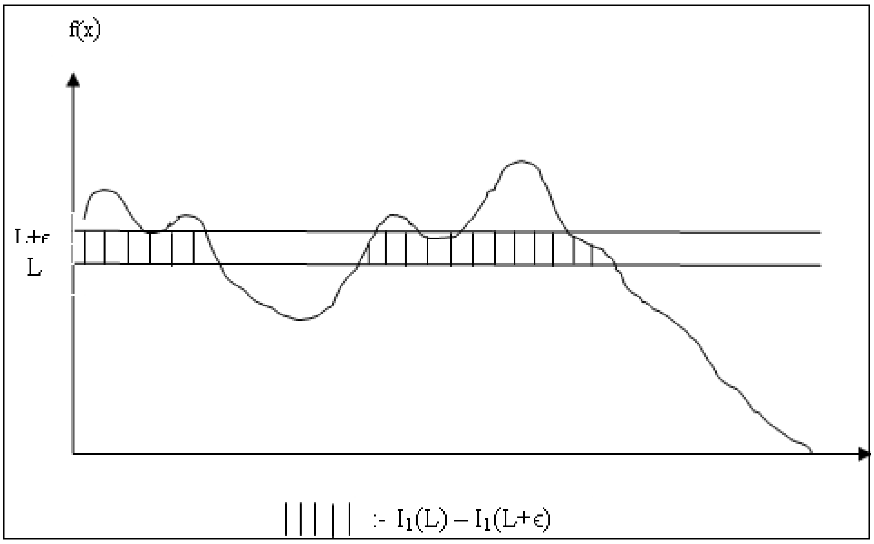

Fig 2: Difference between regions $I_1(L)$ and $I_1(L+\epsilon)$

Lemma-2: If $I_2(L) = \int_{A(L)} (f(x) - L)^2 dx$ then

$$\frac{d}{dL} I_2(L) = -2\int_{A(L)} (f(x) - L)dx \tag{7}$$

Proof: We have,

$$\frac{d}{dL} I_2(L) = lim_{\epsilon\to 0} \frac{\int_{A(L+\epsilon)}(f(x)-(L+\epsilon))^2 dx - \int_{A(L)}(f(x)-L)^2 dx}{\epsilon} \tag{8}$$

Consider,

$I_2(L) - I_2(L+\epsilon)$ $(> 0, \because A_{L+\epsilon} \subset A_L)$

$= \int_{A(L)} (f(x) - L)^2 dx - \int_{A(L+\epsilon)} (f(x) - (L+\epsilon))^2 dx$

$= \int_{A(L)} (f(x) - L)^2 dx - \int_{A(L+\epsilon)} \{(f(x) - L)^2 - 2\epsilon(f(x) - L) + \epsilon^2\} dx$

$= \int_{A(L)} (f(x) - L)^2 dx - \int_{A(L+\epsilon)} (f(x) - L)^2 dx + 2\epsilon \int_{A(L+\epsilon)} (f(x) - L)dx - \epsilon^2 \int_{A(L+\epsilon)} dx$

$= \int_{A(L)-A(L+\epsilon)} (f(x) - L)^2 dx + 2\epsilon \int_{A(L+\epsilon)} (f(x) - L)dx - \epsilon^2 \int_{A(L+\epsilon)} dx$

$(\because A_{L+\epsilon} \subset A_L)$

$$\leq \int_{A(L)-A(L+\epsilon)} \epsilon^2 dx + 2\epsilon \int_{A(L+\epsilon)} (f(x) - L)dx - \epsilon^2 \int_{A(L+\epsilon)} dx \tag{9}$$

$\Longrightarrow$

$$\frac{I_2(L+\epsilon) - I_2(L)}{\epsilon} \geq -\epsilon \int_{A(L)-A(L+\epsilon)} dx - 2\int_{A(L+\epsilon)} (f(x) - L)dx + \epsilon \int_{A(L+\epsilon)} dx \tag{10}$$

From (8) and (10) we have proved that:

$\frac{d}{dL} I_2(L) = -2\int_{A(L)} (f(x) - L)dx$

Remark: If A(L) = {x : f(x) < L} then $\frac{d}{dL} I_2(L) = 2\int_{A(L)} (\mathrm{f(x)} - \mathrm{L})\mathrm{dx}$

3. Determination of PMVRA's

In the following we obtain L^*'s for the three norms d_S, d_1, d_2 and use them to develop the corresponding PMVRA's. That is, we describe the methods of obtaining the modified functions (constants) and the domains for modifications under (a) Sup-norm (b) L1-norm and (c) L2-norm.

3.1. Sup-norm (d_S):

From (2) and (4) we have

$\delta(L) = d_S(f^D, f_1^D) = Maximum\{d_S(f^A, f_1^A), d_S(f^B, f_1^B)\}$.

$d_S(f^A, f_1^A) = Sup_{x \in A(L)}\{|f(x) - L|\} = Sup_{x \in A(L)}\{f(x) - L\} = f(b) - L$ and

$d_S(f^B, f_1^B) = Sup_{x \in B(L)}\{|f(x) - L|\} = Sup_{x \in B(L)}\{L - f(x)\} = L - f(a)$

Hence,

$d_S(f^D, f_1^D) = Maximum\{f(b) - L, L - f(a)\}$

$= Maximum\{(L - f(a)), (f(b) - f(a)) - (L - f(a))\}$

$= \delta_S(L) \geq (f(b) - f(a))/2$

(= a minimum value attained by $\delta_S(L)$ for $f(a) \leq L \leq f(b)$).

$= \delta_S(L^*)$, where

$$L^* = (f(b) + f(a))/2 \tag{11}$$

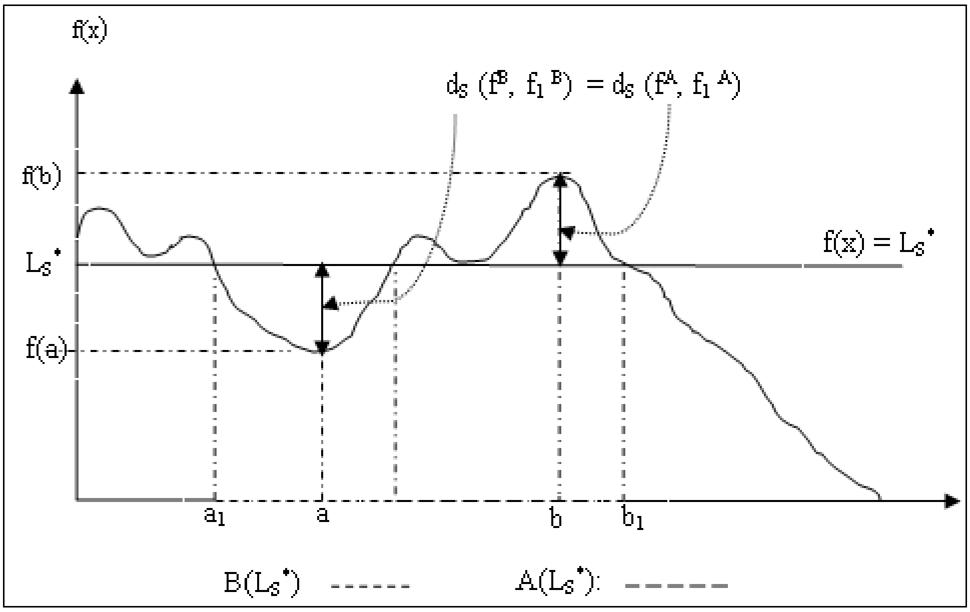

Fig 3: Optimal L corresponding to Sup-norm

3.2. L_1-norm (d_1)

In this case, from (2) and (5) we have

$\delta_1(L) = d_1(f^D, f_1^D) = d_1(f^A, f_1^A) + d_1(f^B, f_1^B)$

$d_1(f^A, f_1^A) = \int_{A(L)}\{|f(x) - L|\}dx = \int_{A(L)}(f(x) - L)dx$ and

$d_1(f^B, f_1^B) = \int_{B(L)}\{|f(x) - L|\}dx = \int_{B(L)}(L - f(x))dx$ Hence,

Since, the sets A(L) and B(L) are decreasing and increasing (in L), respectively, the integral $\int_{A(L)}(f(x) - L)dx$ is decreasing while the integral $\int_{B(L)}(L - f(x))dx$ is increasing and their respective derivatives w. r. t. L are $-\Lambda\{A(L)\}$ and $\Lambda\{B(L)\}$, where $\Lambda(.)$ is the Lebesgue measure (from Lemma-1).

Hence, the differentiation of $\delta_1(L)$ w. r. t. L is given by

$\frac{d}{dL}\delta_1(L) = -\Lambda\{A(L)\} + \Lambda\{B(L)\}$ and $\frac{d}{dL}\delta_1(L) = 0$ gives L (= L*, say) such that .

$$\Lambda\{A(L)\} = \Lambda\{B(L)\} \tag{12}$$

Also it can be observed that, $\Lambda\{A(L)\}$ is decreasing and $\Lambda\{B(L)\}$ is increasing in L, $\delta_1(L)$ is minimum at L^*, where L^* is such that $\Lambda\{A(L^*)\} = \Lambda\{B(L^*)\}$. The value of L^* corresponding to L_1-norm is shown below in Figure-4.

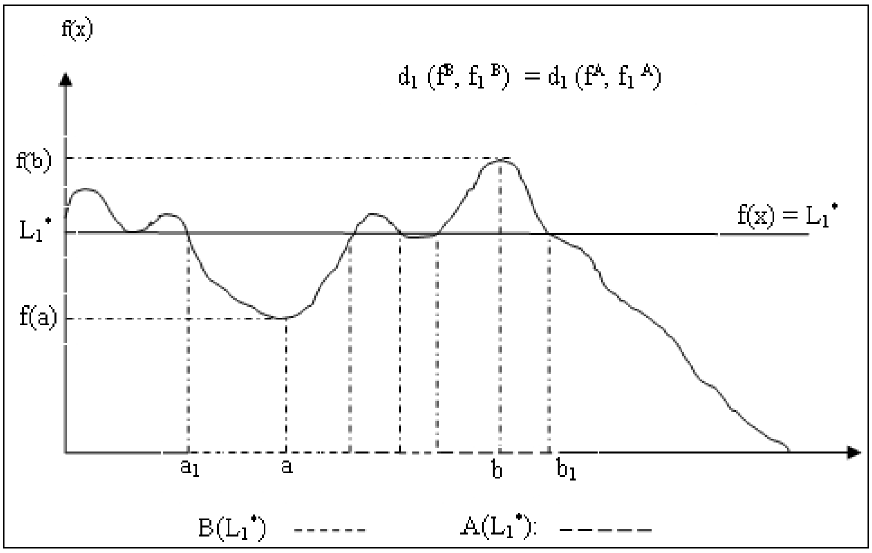

Fig 4: Optimal L corresponding to L_1-norm

3.3. L_2-norm (d_2)

In this case, from (2) and (5) we have

$$\delta_2(L) = d_2(f^D, f_1^D) = d_2(f^A, f_1^A) + d_2(f^B, f_1^B)$$

$$d_2(f^A, f_1^A) = \int_{A(L)} (f(x) - L)^2 dx \quad \text{and} \quad d_2(f^B, f_1^B) = \int_{B(L)} (f(x) - L)^2 dx$$

Differentiating $\delta_2(L)$ w. r. t. L we have

$$\frac{d}{dL}\delta_2(L) = \frac{d}{dL}\int_{A(L)} (f(x) - L)^2 dx + \frac{d}{dL}\int_{B(L)} (f(x) - L)^2 dx$$

From Lemma-2,

$$\frac{d}{dL}\int_{A(L)} (f(x) - L)^2 dx = -2\int_{A(L)} (f(x) - L)dx \quad \text{and}$$

$$\frac{d}{dL}\int_{B(L)} (f(x) - L)^2 dx = -2\int_{B(L)} (f(x) - L)dx$$

$\therefore \frac{d}{dL}\delta_2(L) = 0$ gives L (= L*, say) such that .

$$\int_{A(L^*)} (f(x) - L^*)dx = \int_{B(L^*)} (L^* - f(x))dx \tag{13}$$

Also it can be observed that,

$\int_{A(L)} (f(x) - L)dx$ (= Area{A(L), f}, say) is decreasing and $\int_{B(L)} (L - f(x))dx$ (= Area{B(L), f}, say) is increasing in L, $\delta_2(L)$) is minimum at L^* , L^* is such that,

$$\text{Area}\{A(L^*), f\} = \text{Area}\{B(L^*), f\} \tag{14}$$

The value of L^* corresponding to L_2-norm is shown below in Figure-5.

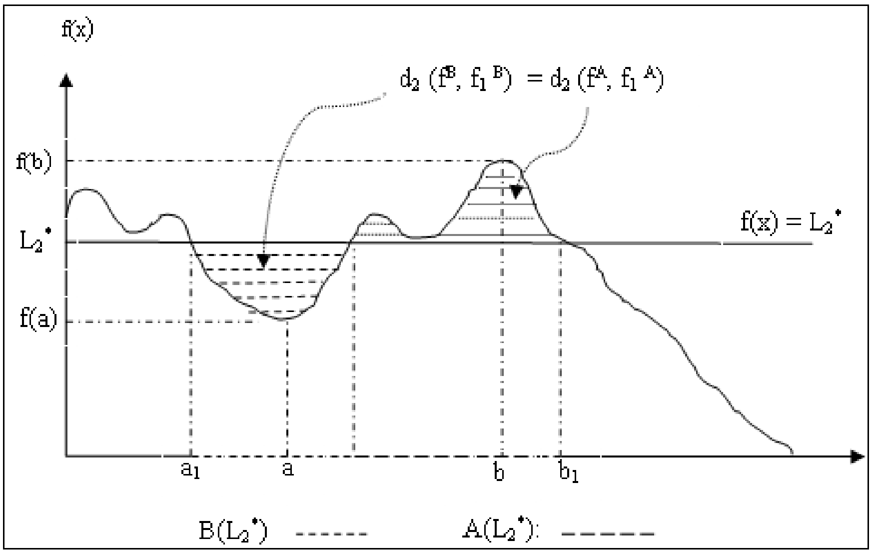

Fig 5: Optimal L corresponding to L_2-norm

4. Comments and Remarks

i). Performance of PMVRA

Note that, removal of violation only on [u, v] may result in violation at u and v. But, in the proposed algorithm, we remove violation over [u, v] along with rectification of function over a super set of [u, v]. As such, in PMVRA an iteration removes at least one turning point, and hence the number of iterations to attain the non-increasing property will be lesser than the number of turning points.

ii). Termination of PMVRA

To ensure the termination of PMVRA, we assume that f(.) has k, a finite number of turning points. The PMVRA identifies the interval of maximum violation (if any) and on a certain super set of this interval the function is modified by a suitable constant that depends on the choice of the norm. In the subsequent stage, modification if required will be on a domain excluding the interval of maximum violation. As such, after each modification the domain of the function that needs to be considered reduces very significantly. It is to be noted that there are at most (k - 1) violating intervals for $f_1(.)$. Hence, the algorithm requires at most k iterations.

5. Conclusion

In this research paper have described an algorithm for obtaining closest non-increasing function to a given one. We have obtained closest constant value on superset D of maximum violation region when measure of closeness is Sup-norm, L_1-norm and L_2-norm.

Equation (11) gives the explicit value of L^* in case of Sup-norm.

We have made an application of Lemma–1 and Lemma–2 while determining closest L^* to a given function on superset D of maximal violation region under the closeness measures L_1-norm and L_2-norm, respectively.

Equation (12) gives the value of L^* in case of L_1-norm. $\delta_1(L)$ is the absolute deviation of f(X) from L therefore, if we assume X has uniform distribution on D then, L^* which minimizes $\delta_1(L)$ is a median of f(X) on D.

Equation (13) and (14) gives the value of L^* in case of L_2-norm. $\delta_2(L)$ is the mean square deviation of f(X) from L therefore, if we assume X has uniform distribution on D then, L^* which minimizes $\delta_2(L)$ is an arithmetic mean of f(X) on D.

When we came across the problem of density estimation and density function is known to be non-increasing as we move away from its modal point. In such situation, application of PMVRA can be made on densiy estimator to obtain closest non-decreasing or non-increasing function to a given density estimator. Closeness can be measured on the basis of given norm such as Sup-norm, L_1-norm and L_2-norm.

One can provide an algorithm for obtaining closest non-increasing function to a given one when closeness measure is different from the above discussed norms. Also, for computation purpose, one can develope computer softwares to impliment PMVRA for different closeness measures.

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