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On the Inequality for Arctangent Function

***1Sumedh Bhaskarrao Thool**

*1Department of Mathematics, Government Vidarbha Institute of Science and Humanities, Amravati, Maharashtra, India.

Abstract

The arctangent function is commonly used in solving problems involving angles in trigonometry, as well as in applications like robotics, computer graphics, and signal processing. It provides a way to convert between angular measures and the ratios of sides in right triangles, making it a fundamental tool in mathematics and science. This research paper provides a sharp lower bound for the arctangent function $\tan^{-1} x$. Also, at the same time, upper bound of same kind obtained, and hence obtained a double inequality.

Keywords: Trigonometric functions, Inverse trigonometric functions, arctangent function, Masjed-Jamei inequality

Introduction

Masjed-Jamei[5] established an inequality given by (1.1), in his paper, he proposed an upper bound of special type which is a ratio of $x \cdot \sinh^{-1} x$ and $\sqrt{1+x^2}$. In [3], Chesneau and Bagul corroborated reverse trigonometric inequality in an attempt of obtaining lower bound. By combining the upper and lower bounds in [5] and [3] respectively, we get the following inequality

$$\left(\frac{\sin x \cdot \sinh^{-1} x}{\sqrt{1+x^2}}\right)^{1/2} < \tan^{-1} x < \left(\frac{x \cdot \sinh^{-1} x}{\sqrt{1+x^2}}\right)^{1/2} \tag{1.1}$$

Also, the simple efficient bounds provided for arctangent function by Bagul and Dhaigude[2], depicted in the following:

Theorem 1([2], Theorem 4): *Let $x \in (0, \lambda)$ where $\lambda \in (0, \infty)$. Then the function $g(x) = \frac{\ln\left(\frac{x}{\tan^{-1} x}\right)}{\ln(1+ax^2)}$ is strictly increasing if $a \geq 1$. In particular, we have following several inequalities*

$$x(1+ax^2)^{\eta} < \tan^{-1} x < x(1+ax^2)^{\xi} \tag{1.2}$$

with the best possible constants $\eta = \frac{\ln\left(\frac{\tan^{-1}\lambda}{\lambda}\right)}{\ln(1+a\lambda^2)}$ and $\xi = -1/3a$.

In their paper, they have claimed that the sharpest inequality of the kind (1.2) is observed at $a = 1$ which is given below.

$$x(1+x^2)^{-1/2} < \tan^{-1} x < x(1+x^2)^{-1/3}; x > 0 \tag{1.3}$$

In this paper, the refined bounds are provided which refines the lower bounds given by inequalities (1.1) and (1.3). The detailed development and discussion will be given in the next section.

Main Result and Discussion

Theorem 2: The following inequality holds true

$$x\left(1+\frac{3x^2}{\pi^2}\right)^{b_1} < \tan^{-1}x < x\left(1+\frac{3x^2}{\pi^2}\right)^{b_2};\ x>0 \tag{2.1}$$

for the best possible constants $b_2 = -\pi/9$ *and* $b_1 = \frac{\ln\left(\frac{\tan^{-1}\alpha}{\alpha}\right)}{\ln\left(1+\frac{3\alpha^2}{\pi^2}\right)}$ *where* $\alpha \in (0,\infty)$.

To prove above theorem, we need some lemma that we have discussed below.

First we write down the highly used and utilized lemma in the field of mathematical inequalities i.e. L'Hopital Rule of Monocity, whose statement is as follows:

Lemma 1[1, 8]**:** *Let* $r, s: [m,n] \to \mathbb{R}$ *be two continuous functions which are differentiable on* (m,n) *and* $s' \neq 0$ *in* (m,n). *If* r'/s' *is increasing(or decreasing) on* (m,n), *then the functions* $\frac{r(x)-r(m)}{s(x)-s(m)}$ *and* $\frac{r(x)-r(n)}{s(x)-s(n)}$ *are also increasing(or decreasing) on* (m,n). *If* r'/s' *is strictly monotone, then the monotonicity in the conclusion is also strict.*

Lemma 2: *The real valued function*
$\kappa_6(x) = -\pi x^2 - 3x^4 + 2\pi(1+x^2)^2(\tan^{-1}x)^2 - (\tan^{-1}x)(\pi x + (3\pi-3)x^3 + 3x^5)$ *is negative decreasing on* $(0,\infty)$.

Proof: Differentiating five times the above function $\kappa_6(x)$, we get following

$$\kappa_6'(x) = -\frac{15x^5 + (5\pi+9)x^3 + 3\pi x}{1+x^2} + 8\pi x(1+x^2)(\tan^{-1}x)^2 + (3\pi + (9-5\pi)x^2 - 15x^4)(\tan^{-1}x),$$

$$\kappa_6''(x) = -\frac{60x^6 + (10\pi+90)x^4 + (14\pi+18)x^2}{(1+x^2)^2} + 8\pi(1+3x^2)(\tan^{-1}x)^2 + ((18+6\pi) - 60x^3)(\tan^{-1}x),$$

$$\kappa_6^{(3)}(x) = -\frac{180x^7 + (462-6\pi)x^5 + 348x^3 + (22\pi+18)x}{(1+x^2)^3} + 48\pi x(\tan^{-1}x)^2 + \frac{((22\pi+18) + (54\pi-162)x^2 - 180x^4)(\tan^{-1}x)}{1+x^2},$$

And

$$\kappa_6^{(4)}(x) = -\frac{360x^8 + (1320-48\pi)x^6 + (1752-160\pi)x^4 + (1080-208\pi)x^2}{(1+x^2)^4} - \frac{(360x^5 + (720-96\pi)x^3 + (360-160\pi)x)(\tan^{-1}x)}{(1+x^2)^2} + 48\pi(\tan^{-1}x)^2,$$

$$\kappa_6^{(5)}(x) = -\frac{x}{(1+x^2)^3}\cdot\kappa_7(x),$$

Where

$$\kappa_7(x) = \frac{360x^8 + 1680x^6 + 3072x^4 + (192\pi+1968)x^2 - 576\pi + 2520}{(1+x^2)^2} + \frac{(360x^6 + 1080x^4 + 360 - 256\pi)\cdot\tan^{-1}x}{x},$$

By doing elementary calculations and by using $\tan^{-1}x < x$ in $\kappa_7(x)$, which results into a polynomial with positive coefficients as follows

$$\kappa_8(x) = 360x^{10} + 2160x^8 + 5280x^6 + (6672-256\pi)x^4 + (3768-320\pi)x^2 + 2880 - 832\pi,$$

Since, every coefficient of above polynomial $\kappa_8(x)$ is positive, the graph of it lies above x-axis in the plane and hence it is positive on $(0,\infty)$ i.e. $\kappa_8(x) > 0, \forall x \in (0,\infty)$.

$\therefore\ \kappa_7(x) > 0, \forall x \in (0,\infty)$.

By using elementary calculus, we have

$$\kappa_6^{(i)} < 0, \forall x \in (0,\infty), where\ i = 1,2,3,4,5.$$

Hence, $\kappa_6(x)$ is negative decreasing on $(0,\infty)$.

Lemma 3: *The function* $\kappa(x) = \frac{\ln\left(\frac{\tan^{-1}x}{x}\right)}{\ln\left(1+\frac{3x^2}{\pi^2}\right)}$ *is strictly decreasing on* $(0,\infty)$.

Proof: Denote $\kappa_1(x) = \ln\left(\frac{\tan^{-1}x}{x}\right)$ and $\kappa_2(x) = \ln\left(1+\frac{3x^2}{\pi^2}\right)$ and the right hand limits at 0 are given by $\kappa_1(0+) = 0 = \kappa_2(0+)$.

Differentiating these functions, we get

$$\frac{\kappa_1'(x)}{\kappa_2'(x)} = \frac{1}{6}\kappa_3(x),$$

Where $\kappa_3(x) = \frac{(\pi+3x^2)(x-(1+x^2)\tan^{-1}x)}{x^2(1+x^2)\tan^{-1}x} := \frac{\kappa_4(x)}{\kappa_5(x)}$ say.

We will show the numerator N_1 of $\kappa_3'(x)$ is negative, which will result in $\kappa_3(x)$ is decreasing on $(0,\infty)$.

Now, $N_1(x) = \kappa_5 \cdot \kappa_4' - \kappa_4 \cdot \kappa_5' = x \cdot \kappa_6(x)$, where

$$\kappa_6(x) = -\pi x^2 - 3x^4 + 2\pi(1+x^2)^2(\tan^{-1}x)^2 - (\tan^{-1}x)(\pi x + (3\pi-3)x^3 + 3x^5),$$

So, $N_1(x) < 0$, only if $\kappa_6(x) < 0$, because $x > 0$ as $x \in (0,\infty)$.

By Lemma 2, $\kappa_6(x)$ is negative decreasing on $(0,\infty)$.

$\therefore N_1(x) < 0, \forall x \in (0,\infty)$
$\therefore \kappa_3'(x) < 0, \forall x \in (0,\infty)$

Hence, in $\kappa_3(x)$ is decreasing on $(0,\infty)$.

So, $\frac{\kappa_1'(x)}{\kappa_2'(x)}$ is decreasing on $(0,\infty)$.

Therefore, by L'Hopital Rule of Monocity given in Lemma 1, we have $\frac{\kappa_1(x)-\kappa_1(0+)}{\kappa_2(x)-\kappa_2(0+)}$ is also decreasing on $(0,\infty)$. But, $\kappa_1(0+) = 0 = \kappa_2(0+)$ is already with us.

So, $\frac{\kappa_1(x)}{\kappa_2(x)} = \kappa(x) = \frac{\ln\left(\frac{\tan^{-1}x}{x}\right)}{\ln\left(1+\frac{3x^2}{\pi^2}\right)}$ is decreasing on $(0,\infty)$.

Proof of Theorem 2: To prove the inequality, we form a centralized function $\kappa(x) = \frac{\ln\left(\frac{\tan^{-1}x}{x}\right)}{\ln\left(1+\frac{3x^2}{\pi^2}\right)}$, say.

Now, we must show that this function is strictly decreasing on $(0,\infty)$. So, by above Lemma 3, the function $\kappa(x) = \frac{\ln\left(\frac{\tan^{-1}x}{x}\right)}{\ln\left(1+\frac{3x^2}{\pi^2}\right)}$ is strictly decreasing on $(0,\infty)$.

Hence, for $\alpha \in (0,\infty)$, we have $\kappa(0+) > \kappa(x) > \kappa(\alpha-)$.

The proof of theorem completes with the limits, $b_2 = \kappa(0+) = -\pi/9$ and $b_1 = \kappa(\alpha-) = \frac{\ln\left(\frac{\tan^{-1}\alpha}{\alpha}\right)}{\ln\left(1+\frac{3\alpha^2}{\pi^2}\right)}$.

Remark: For $\alpha = \pi$, we get the following inequality

$$x\left(1+\frac{3x^2}{\pi^2}\right)^{\frac{\ln\left(\frac{\tan^{-1}\pi}{\pi}\right)}{\ln(1+3\pi)}} < \tan^{-1}x < x\left(1+\frac{3x^2}{\pi^2}\right)^{-\pi/9} ;\pi > x > 0 \qquad (2.2)$$

This inequality is of our special interest.

Conclusion

The inequality obtained in (2.1) provides an upper bound which is uniformly sharper than that of the upper bound in (3.1) on $(0, \infty)$. Hence, we disprove the claim in [2]. The lower bound in inequality (2.1), precisely given in (2.2) refines the lower bound given by Masjed-Jamei inequality (1.1) on $(0, \pi)$, also it refines the lower bound in (1.3) on $(0, \pi)$.

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