Volume: 1 Issue: 12 Pages: 26-27

International Journal of Research in Academic World

E-ISSN: 2583-1615
Impact Factor: 4.714

Received: 11/July/2022 Int. J Res. Acad. World. 2022; 1(12):26-27 Accepted: 19/August/2022

A Brief Study on Complex Analysis

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Abstract

Complex analysis, traditionally known as the theory of functions of a complex variable, is the branch of mathematical analysis that investigates functions of complex numbers. Complex analysis is a beautiful, tightly integrated subject. It revolves around complex analytic functions. These are functions that have a complex derivative. Unlike calculus using real variables, the mere existence of a complex derivative has strong implications for the properties of the function. Complex analysis is a basic tool in many mathematical theories. By itself and through some of these theories it also has a great many practical applications.

Keywords: Complex number, limits, continuity, differentiable function, analytic function, CR equation

Introduction

Consider the quadratic equation $x^2+1=0$ and solving this we get $x=\pm\sqrt{-1}$ which is not real. Euler was first to introduce the symbol i, where $i=\sqrt{-1}$ which has the property i2=-1. The surprising fact about complex numbers is that complex numbers arose from the need to solve cubic equation and not quadratic equations. Complex numbers were first introduced by an Italian Mathematician Gerolamo cardano during his first attempts to solve cubic equation of the type $x^3=px+q$ in the 16th century.

Definitions

Complex Number: A number of the form a+ib, where a and b are real numbers and i is called imaginary unit having the property $i^2=-1$ is called complex number.

OR

An ordered pair of real numbers such as (x,y) is called a complex number. If we write

Z=(x,y) or x+iy, where $i=\sqrt{-1}$ is imaginary unit then x is called real part of z and y is called imaginary part of z and are denoted by x=Re(z) and y=Im(z).

Complex Function: A complex function is a function from complex numbers to complex numbers.

In other words,

It is a function that has a subset of the complex numbers as a domain and the complex numbers as a codomain. Complex functions are generally assumed to have a domain that contains a nonempty open subset of the complex plane.

For any complex function, the values z from the domain and their images f(z) in the range may be separated into real and imaginary parts

$$z= x+iy \text{ and } f(z) =f(x+iy) =u(x,y)+iv(x,y)$$

Where x,y, u(x,y), v(x,y) are the real valued function.

Complex analysis, traditionally known as the theory of functions of a complex variable, is the branch of mathematical analysis that investigates functions of complex numbers. It is helpful in many branches of mathematics, including algebraic geometry, number theory, analytic combinatorics, applied mathematics; as well as in physics, including the branches of hydrodynamics, thermodynamics, quantum mechanics, and twistor theory. By extension, use of complex analysis also has applications in engineering fields such as nuclear, aerospace, mechanical and electrical engineering.

As a differentiable function of a complex variable is equal to its Taylor series (that is, it is analytic), complex analysis is particularly concerned with analytic functions of a complex variable (that is, holomorphic functions).

Analytic Function: A function f (z) of the complex variable z is analytic at a point z_0 if its derivative exists not only at z_0 but each point z in some neighborhood of z_0

A function f (z) is said to be analytic in a domain D, if it is analytic at every point in D.

Analytic functions are also known as regular function or holomorphic function.

If a function f (z) fails to be analytic at a point z_0, then z_0 is called a singular point or singularity of the function.

Cauchy Riemann Equation: If a function f(z)=u(x,y)+iv(x,y) is analytic in domain D, the partial derivatives u_x, v_x, u_y, v_y should exists and satisfy the equations $u_x = v_y, u_y = -v_x$.

$$\text{Or } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Example 1: Verify Cauchy-Riemann equation for $f(z) = z^2$

Solution: $f(z) = z^2 = (x + iy)^2$

$$= =x^2 + i^2y^2 + 2xiy$$

$$= (x^2 - y^2) + i2xy$$

$$\therefore \quad u = x^2 - y^2 \quad v = 2xy$$

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$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 2y.$$

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial y} = 2x$$

We have $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Hence C-R equations are satisfied.

Example 2: Verify Cauchy-Riemann equation for $f(z) = e^x \cos y + ie^x \sin y$

Solution: $u = e^x \cos y$, $v = e^x \sin y$

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial v}{\partial x} = e^x \sin y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y \qquad \frac{\partial v}{\partial y} = e^x \cos y$$

We have $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Hence C-R equations are satisfied.

Every holomorphic function can be separated into its real and imaginary parts f(x+iy)=u(x,y)+iv(x,y) and each of these is a harmonic function on R2 (each satisfies Laplace's equation $\Delta^2 u = \Delta^2 v = 0$)

Laplace Equation in Two Dimension:

The equation $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ is called Laplace equation in two dimension.

Harmonic Function: Harmonic function is a twice continuously differentiable function. i.e.

A function $u(x, y)$ which possesses continuous partial derivatives of the *first and second order and satisfies the Laplace equation* is called a harmonic function.

If u is harmonic then v is the harmonic conjugate of u

Example 1: Show that $u = 2xy - 2x + y$ is harmonic

Solution: $u_x = 2y - 2$ $u_y = 2x + 1$

$u_{xx} = 0 \quad u_{yy} = 0$

$\therefore u_{xx} + u_{yy} = 0$

$\therefore u(x, y)$ *is harmonic*.

Example 2: Show that $u = x^3 - 3xy^2$ is harmonic and find its harmonic conjugates

Solution: Given that $u = x^3 - 3xy^2$

$u_x = 3x - 3y^2$ and $u_y = -6xy$

$u_{xx} = 6x \quad u_{yy} = -6x$

$\therefore u_{xx} + u_{yy} = 0$

$\therefore u(x, y)$ *is harmonic*.

Let $v(x, y)$ be the harmonic conjugate of $u(x, y)$

Now $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$dv = 6xy\, dx + (3x^2 - 3y^2) dy$

$M = 6xy, \qquad N = 3x^2 - 3y^2$

$$v = \int_{y\ constant} M\, dx + \int (terms\ in\ N\ independent\ of\ x)\, dy$$

$$= \int 6xy\, dx + \int -3y^2 dy$$

$$v = 6y\frac{x^2}{2} + 3\frac{y^3}{3} + c = 3x^2 y - y^3 + c.$$

Applications

One of the central tools in complex analysis is the line integral. The line integral around a closed path of a function that is holomorphic everywhere inside the area bounded by the closed path is always zero, as is stated by the Cauchy integral theorem. The values of such a holomorphic function inside a disk can be computed by a path integral on the disk's boundary (as shown in Cauchy's integral formula). Path integrals in the complex plane are often used to determine complicated real integrals, and here the theory of residues among others is applicable (see methods of contour integration). A "pole" (or isolated singularity) of a function is a point where the function's value becomes unbounded, or "blows up". If a function has such a pole, then one can compute the function's residue there, which can be used to compute path integrals involving the function; this is the content of the powerful residue theorem. The remarkable behavior of holomorphic functions near essential singularities is described by Picard's theorem. Functions that have only poles but no essential singularities are called meromorphic Laurent series are the complex-valued equivalent to Taylor series, but can be used to study the behavior of functions near singularities through infinite sums of more well understood functions, such as polynomials.

Conclusion

Complex analysis has wide applications in different fields. Complex analysis is used in major areas in engineering, signal processing and control theory, especially in the analysis of stability of systems and controls design. This is used in circuit theory, electromagnetism and electrostatics in electrical engineering in Fluid Dynamics complex functions are used in modelling of potential flow in two dimension.

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